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On an Identity of Ramanujan Based on The Hypergeometric Series ${}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x)$.

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Dedicated to the Memory of Professor Kermit Sigmon

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and establishing the identity by comparing their Laurent series expansions at a pole.

1. INTRODUCTION

Recently, B. C. Berndt, S. Bhargava and F. Garvan [1] provided the first proof of the following remarkable statement made by Ramanujan.

THEOREM A. *For* $0 \leqslant q < 1$, *let* $a = a(q) = \vartheta_3(q)\,\vartheta_3(q^3) + \vartheta_2(q)\,\vartheta_2(q^3)$, $c = c(q) = \frac{1}{2}a(q^{1/3}) - \frac{1}{2}a(q)$ *and* $h = (c^3/a^3)$; *and define*

$$az = \int_0^{\varphi} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; h \sin^2 t\right) dt. \tag{1.1}$$

Then

$$\varphi = z + 3 \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{n(1 + q^n + q^{2n})}. \tag{1.2}$$

The significance of this seemingly mysterious theorem can easily be understood and appreciated through the following familiar classical analogue:

THEOREM B. *Let* $a = \vartheta_3^2(q)$, $c = \vartheta_2^2(q)$ *and* $k^2 = (c^2/a^2)$. *Define*

$$az = \int_0^{\varphi} {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}; k^2 \sin^2 t\right) dt. \tag{1.3}$$

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Then

$$\varphi = z + 2 \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{n(1+q^{2n})}. \tag{1.4}$$

The proof of Theorem B goes as follows: We recall [4, p. 101]

$${}_2F_1\left(\frac{1}{2}a + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}a; \frac{1}{2}; \sin^2 z\right) = \frac{\cos az}{\cos z}. \tag{1.5}$$

Let $a = 0$ and $x = \sin^2 z$. Then (1.5) becomes

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; x\right) = \frac{1}{\sqrt{1-x}}, \qquad |x| < 1. \tag{1.6}$$

From (1.6), we can write (1.3) as

$$\begin{aligned}\vartheta_3^2 z &= \int_0^{\varphi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; k^2 \sin^2 \theta\right) d\theta \\ &= \int_0^{\varphi} (1 - k^2 \sin^2 \theta)^{-1/2}\, d\theta \\ &= \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\end{aligned}$$

where $\vartheta_3 = \vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$.

Thus, from the definition of $sn(z, k)$ [See 5, p. 492], we conclude that

$$sn(\vartheta_3^2 z, k) = \sin \varphi. \tag{1.7}$$

To solve φ in terms of z, we differentiate (1.7) with respect to z. Then (See [5, p. 511])

$$\frac{d\varphi}{dz} = \vartheta_3^2 \frac{cn(\vartheta_3^2 z)\, dn(\vartheta_3^2 z)}{\cos \varphi} = \vartheta_3^2\, dn(\vartheta_3^2 z) = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n \cos 2nz}{1+q^{2n}}. \tag{1.8}$$

The identity (1.4) now follows immediately by integrating (1.8). This completes the proof of Theorem B.

Our proof of Theorem A, although differing substantially from that of [1], is essentially based on their idea which we now describe: Let

$$S(x) = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x^2).$$

We first observe that it satisfies the following cubic equation

$$4(1-x^2)\, y^3(x)-3y(x)=1. \tag{1.9}$$

To prove this, we choose $a=\frac{1}{3}$ and $z=\sin^{-1} x$ in (1.5). Then $x=\sin z$, $\cos z=\sqrt{1-x^2}$ and

$$S(x)={}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x^2)=(1-x^2)^{-1/2}\cos(\tfrac{1}{3}\sin^{-1} x), \qquad |x|<1.$$

Now (1.9) follows immediately from the identity

$$\cos\theta=4\cos^3\frac{\theta}{3}-3\cos\frac{\theta}{3}.$$

In fact, $S(x)$ is the unique analytic solution of (1.9) satisfying the condition $S(0)=1$, whereas both of the two other solutions have $y(0)=-\frac{1}{2}$, hence are ramified at $x=0$.

To see the connection of the identity (1.9) with Theorem A, we differentiate (1.1) with respect to z. Then

$$a=S(\sqrt{h}\sin\varphi)\,\frac{d\varphi}{dz}.$$

Choosing $y=S(x)$ and $x=\sqrt{h}\sin\varphi$ in (1.9), we obtain

$$4h\sin^2\varphi=4-\left(\frac{1}{a}\frac{d\varphi}{dz}\right)^3-3\left(\frac{1}{a}\frac{d\varphi}{dz}\right)^2. \tag{1.10}$$

Therefore, to establish Theorem A, it is sufficient to show that φ defined as in (1.2) satisfies the Eq. (1.10). To simplify computation further, we differentiate (1.10) once more with respect to z and prove that φ satisfies the following differential equation

$$8\sin\varphi\cos\varphi=-\frac{3}{c^3}\frac{d^2\varphi}{dz^2}\left(2a+\frac{d\varphi}{dz}\right), \tag{1.11}$$

where c is defined in Theorem A.

It is worthwhile to insert a comment about the nature and the domain of the solution of the autonomous differential equation:

$$\varphi'=F(\varphi), \qquad \varphi(0)=0,$$

where

$$F(\varphi) = \frac{a}{S(\sqrt{h}\sin\varphi)}.$$

We note that if $0 \leqslant h < 1$, then there exists a horizontal strip $T = \{\varphi\colon |\mathrm{Im}\,\varphi| < b\}$ containing the entire real axis such that for all $\varphi \in T$, $|\sqrt{h}\sin\varphi| \leqslant 1-\delta$ for some $\delta > 0$. Since $S(z)$ is analytic and zero free in $|z| < 1$, $F(\varphi)$ is analytic in T and

$$M = \sup_{\varphi \in T} |F(\varphi)| < \infty.$$

The key properties of the solutions of this differential equation can be derived from the following well-known theorem [3, p. 34]: Suppose $f(z, w)$ is an analytic complex-valued function on the domain

$$R_2\colon |z - z_0| < a, \, |w - w_0| < b(a, b > 0).$$

Let $M = \sup_{(z, w) \in R_2} |f(z, w)|$, $\alpha = \min(a, (b/M))$. Then there exists on $|z - z_0| < \alpha$ a unique analytic function φ which is a solution of

$$w' = f(z, w)$$

satisfying $\varphi(z_0) = w_0$.

We readily conclude from this theorem that $\varphi(z)$ is analytic in the disc $D_0 = \{z\colon |z| < b/M\}$; moreover, $\varphi(z)$ is real valued if z is confined to the real axis. (This is due to the fact that the above theorem is derived from the method of Picard's successive approximation, since $F(\varphi)$ is real with real initial condition, the solution φ is real on the real axis). We now choose a point x_1 on the real axis such that $|x_1| < b/M$, $x_1 \neq 0$. Repeating the argument above, we see that there exists a unique solution $\varphi_1(z)$ which is analytic in $D_1 = \{z\colon |z - x_1| < b/M\}$ and $\varphi_1(x_1) = \varphi(x_1)$, $\varphi_1(z) = \varphi(z)$ if $z \in D_1 \cap D_0$; therefore φ_1 is the unique analytic continuation of φ to the disc D_1. Clearly, this process can be repeated indefinitely along the real axis using a countable number of discs $\{D_n\}$ of the same radius b/M; thus $\varphi(z)$ is analtyic in $\Omega = \bigcup_{n=0}^{\infty} D_n$. In particular, φ is analytic on a horizontal strip containing the real axis.

The key idea of our proof is to express (1.11) in terms of the familiar theta functions and Weierstrass elliptic function. We then appeal to the following simple fact: If f and g are doubly periodic and the principal parts of their Laurent series expansions at each pole are equal, then $f - g$ is constant; moreover, if this constant is zero, then $f \equiv g$.

Proof of Theorem A. We first express $d\varphi/dz$ in terms of theta functions:

$$\begin{aligned}\frac{d\varphi}{dz} &= 1+6\sum_{n=1}^{\infty}\frac{q^n\cos 2nz}{1+q^n+q^{2n}}\\ &= 1+\frac{3}{2}i\left\{\frac{\vartheta_4'}{\vartheta_4}\left(z-\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)-\frac{\vartheta_4'}{\vartheta_4}\left(z+\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)\right\}\\ &= -2+\frac{3}{2}i\left\{\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi\tau}{2}\,\middle|\,\frac{3\tau}{2}\right)-\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi\tau}{2}\,\middle|\,\frac{3\tau}{2}\right)\right\},\end{aligned} \tag{2.1}$$

where the differentiation is referred to the variable z. And to prove the last two equalities, we recall that [5, p. 489]

$$\begin{aligned}\frac{\vartheta_4'}{\vartheta_4}(x\,|\,\tau) &= 4\sum_{n=1}^{\infty}\frac{q^n\sin 2nx}{1-q^{2n}}\\ &= \frac{2}{i}\sum_{n=1}^{\infty}\frac{q^n(t^n-t^{-n})}{1-q^{2n}}\ (t=e^{2ix} \text{ and } q=e^{\pi i\tau},\ \operatorname{Im}\tau>0).\end{aligned}$$

We now replace q by $q^{3/2}$ and t by $\sqrt{q}\,u$. Then

$$\frac{\vartheta_4'}{\vartheta_4}\left(z+\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)=\frac{2}{i}\sum_{n=1}^{\infty}\frac{q^{2n}u^n-q^nu^{-n}}{1-q^{3n}}\quad\left(u=\frac{1}{\sqrt{q}}t=e^{2iz}\right). \tag{2.2}$$

Similarly, replacing q by $q^{3/2}$ and t by $u/\sqrt{q}$, we obtain

$$\frac{\vartheta_4'}{\vartheta_4}\left(z-\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)=\frac{2}{i}\sum_{n=1}^{\infty}\frac{q^nu^n-q^{2n}u^{-n}}{1-q^{3n}}.$$

Hence

$$\begin{aligned}&\frac{\vartheta_4'}{\vartheta_4}\left(z-\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)-\frac{\vartheta_4'}{\vartheta_4}\left(z+\frac{\pi\tau}{4}\,\middle|\,\frac{3\tau}{2}\right)\\ &\quad=\frac{2}{i}\left(\sum_{n=1}^{\infty}\frac{q^n(u^n+u^{-n})}{1+q^n+q^{2n}}\right)=\frac{4}{i}\sum_{n=1}^{\infty}\frac{q^n\cos 2nz}{1+q^n+q^{2n}}.\end{aligned}$$

This yields the first part of the identity. The second identity is the immediate consequence of the well-known fact that $\vartheta_4(z+(\pi\tau/2)\,|\,\tau)=iq^{-1/4}e^{-iz}\vartheta_1(z\,|\,\tau)(q=e^{\pi i\tau})$. Since it is very straightforward, we omit its proof.

In the followings, we will express various crucial quantities in terms of the Weierstrassian and Jacobian elliptic functions. We find that it is more convenient to base computation on τ (rather than on $3\tau/2$), so, we will redefine our $d\varphi/dz$ to be

$$\frac{d\varphi}{dz}=1+\frac{3i}{2}\left\{\frac{\vartheta_4'}{\vartheta_4}\left(z-\frac{\pi\tau}{6}\,\middle|\,\tau\right)-\frac{\vartheta_4'}{\vartheta_4}\left(z+\frac{\pi\tau}{6}\,\middle|\,\tau\right)\right\}$$

$$=-2+\frac{3i}{2}\left\{\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi\tau}{3}\,\middle|\,\tau\right)-\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi\tau}{3}\,\middle|\,\tau\right)\right\}. \tag{2.3}$$

When there is no confusion, we will write $\vartheta_i(z)=\vartheta_i(z\mid\tau)$, $i=1,4$. And we note that with the replacement of τ by $3\tau/2$, the identity (2.3) becomes (2.1).

Recall that

$$\wp(z)=-\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(z)-\frac{1}{3}P, \tag{2.4}$$

where

$$P=1-24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}, \tag{2.5}$$

and $\wp(z)$ is the Weierstrassian elliptic function of periods π and $\pi\tau$. Since both sides are doubly periodic functions, this identity can be established easily by comparing the Laurent series expansions at the poles.

Therefore, from (2.3),

$$\frac{d^2\varphi}{dz^2}=\frac{3i}{2}\left(\wp\left(z-\frac{\pi\tau}{3}\right)-\wp\left(z+\frac{\pi\tau}{3}\right)\right). \tag{2.6}$$

And again from (2.3), we see that

$$e^{i\varphi}=ie^{-2iz}\left(\frac{\vartheta_1(z-\pi\tau/3)}{\vartheta_1(z+\pi\tau/3)}\right)^{3/2}$$

and

$$e^{-i\varphi}=-ie^{2iz}\left(\frac{\vartheta_1(z+\pi\tau/3)}{\vartheta_1(z-\pi\tau/3)}\right)^{3/2}.$$

Hence

$$\sin\varphi=\frac{1}{2}\left\{e^{2iz}\left(\frac{\vartheta_1(z+\pi\tau/3)}{\vartheta_1(z-\pi\tau/3)}\right)^{3/2}+e^{-2iz}\left(\frac{\vartheta_1(z-\pi\tau/3)}{\vartheta_1(z+\pi\tau/3)}\right)^{3/2}\right\}$$

and

$$\sin\varphi\cos\varphi = \frac{1}{4i}\left\{e^{4iz}\left(\frac{\vartheta_1(z+\pi\tau/3)}{\vartheta_1(z-\pi\tau/3)}\right)^3 - e^{-4iz}\left(\frac{\vartheta_1(z-\pi\tau/3)}{\vartheta_1(z+\pi\tau/3)}\right)^3\right\}. \tag{2.7}$$

It is worthwhile to note that due to the presence of the radicals, $\sin\varphi$ is not elliptic; however, $\sin^2\varphi$ and $\sin\varphi\cos\varphi$ are both elliptic functions.

Our goal is to prove (1.11) and from (2.3), (2.6) and (2.7), we see that (1.11) is equivalent to the following interesting identity:

$$\begin{aligned} &e^{4iz}\left(\frac{\vartheta_1(z+\pi\tau/3)}{\vartheta_1(z-\pi\tau/3)}\right)^3 - e^{-4iz}\left(\frac{\vartheta_1(z-\pi\tau/3)}{\vartheta_1(z+\pi\tau/3)}\right)^3 \\ &\quad = \frac{9}{4c^3(q^{2/3})}\left(\wp\left(z-\frac{\pi\tau}{3}\right) - \wp\left(z+\frac{\pi\tau}{3}\right)\right) \\ &\qquad \times\left\{2a(q^{2/3}) - 2 + \frac{3i}{2}\left[\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi\tau}{3}\right) - \frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi\tau}{3}\right)\right]\right\}. \end{aligned} \tag{2.8}$$

We observe that both sides of the above identity are elliptic functions of periods π and $\pi\tau$, and both are equal to zero at $z=0$. Moreover, the poles are of order 3; therefore, to prove this identity, it is sufficient to show that the coefficients of the Laurent series expansions corresponding to the terms $(z-\pi\tau/3)^{-n}$, $n=1$, 2 and 3, are equal on both sides. In the followings, we will deal with the expansion of $z=\pi\tau/3$ only, the expansion at $z=-\pi\tau/3$ can be obtained in the identical fashion.

To make computation easier, we will make the change of variable $t=z-\pi\tau/3$. Then (2.8) becomes

$$\begin{aligned} &q^{4/3}e^{4it}\left(\frac{\vartheta_1(t+2\pi\tau/3)}{\vartheta_1(t)}\right)^3 - q^{-4/3}e^{-4it}\left(\frac{\vartheta_1(t)}{\vartheta_1(t+2\pi\tau/3)}\right)^3 \\ &\quad = \frac{9}{4c^3(q^{2/3})}\left(\wp(t) - \wp\left(t+\frac{2\pi\tau}{3}\right)\right)\Bigg(2a(q^{2/3}) - 2 \\ &\qquad + \frac{3i}{2}\left(\frac{\vartheta_1'}{\vartheta_1}\left(t+\frac{2\pi\tau}{3}\right) - \frac{\vartheta_1'}{\vartheta_1}(t)\right)\Bigg). \end{aligned} \tag{2.9}$$

We now write out the Laurent series expansions at $t=0$ of various quantities in (2.9):

$$\wp(t) = \frac{1}{t^2} + O(t^2) \qquad \text{(See [5, p. 436])}$$

$$\wp\left(t + \frac{2\pi\tau}{3}\right) = \wp\left(\frac{2\pi\tau}{3}\right) + \wp'\left(\frac{2\pi\tau}{3}\right) t + \cdots \tag{2.10}$$

$$\frac{\vartheta_1'}{\vartheta_1}(t) = \frac{1}{t} - \frac{1}{3} Pt + \cdots \qquad \text{(From(2.10) and (2.4))}$$

$$\frac{\vartheta_1'}{\vartheta_1}\left(t + \frac{2\pi\tau}{3}\right) = \frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\right) + \left(\frac{\vartheta_1'}{\vartheta_1}\right)'\left(\frac{2\pi\tau}{3}\right) t + \cdots$$

Then the series expansion of the right-hand side of (2.9) is

$$\frac{9}{4c^3(q^{2/3})}\left\{-\frac{3i}{2}\frac{1}{t^3} + \left(2a(q^{2/3}) - 2 + \frac{3i}{2}\frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\right)\right)\frac{1}{t^2} + \frac{3}{2}i\left(\wp\left(\frac{2\pi\tau}{3}\right) + \left(\frac{\vartheta_1'}{\vartheta_1}\right)'\left(\frac{2\pi\tau}{3}\right) + \frac{1}{3}P\right)\frac{1}{t} + \cdots\right\} \tag{2.11}$$

We note that from (2.4), $\wp(z) = -(\vartheta_1'/\vartheta_1)'(z) - \frac{1}{3}P$, the coefficient of $1/t$ in the above series is zero.

To find out the power series of the left-hand side of (2.9), we write

$$\vartheta_1(t) = \vartheta_1'(0)\, t + \frac{\vartheta_1'''(0)}{6} t^3 + \cdots$$

$$\vartheta_1\left(t + \frac{2\pi\tau}{3}\right) = \vartheta_1\left(\frac{2\pi\tau}{3}\right) + \vartheta_1'\left(\frac{2\pi\tau}{3}\right) t + \vartheta_1''\left(\frac{2\pi\tau}{3}\right)\frac{t^2}{2} + \cdots$$

and

$$e^{4it} = 1 + 4it - 8t^2 + \cdots$$

Then the left-hand side of (2.8) it has the following series expansion at $t = 0$:

$$q^{4/3}\alpha^3\left\{\frac{1}{t^3} + \left(3\frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\right) + 4i\right)\frac{1}{t^2} + \frac{0}{t} + \cdots\right\} \tag{2.12}$$

where

$$\alpha = \frac{\vartheta_1(2\pi\tau/3)}{\vartheta_1'(0)}.$$

We remark that the coefficient corresponding to the term $1/t$ is zero. This is due to the fact that the function $e^{4it}(\vartheta_1(t + 2\pi t/3)/\vartheta_1(t))^3$ is elliptic, of periods

π and $\pi\tau$ (This is easily verified using the properties that $\vartheta_1(t+\pi\tau) = -q^{-1}e^{-2it}\vartheta_1(t)$ and $\vartheta_1(t+\pi) = -\vartheta_1(t)$), and $t=0$ is the only pole in each period parallelogram, therefore the residue at $t=0$ must be zero.

We see that to establish (2.8) we need to prove that the coefficients in (2.11) and (2.12) are equal. That is,

$$q^{4/3}\alpha^3 = \frac{-27i}{8c^3(q^{2/3})} \tag{2.13}$$

and

$$q^{4/3}\alpha^3\left(3\frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\right)+4i\right) = \frac{9}{4c^3(q^{2/3})}\left(2a(q^{2/3})-2+\frac{3i}{2}\frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\right)\right) \tag{2.14}$$

We now establish (2.13). It is the same as proving that $q^{4/9}\alpha = (3i/2c(q^{2/3}))$ or equivalently (after replacing τ by $(3\tau/2)$)

$$q^{2/3}\frac{\vartheta_1(\pi\tau \mid 3\tau/2)}{\vartheta_1'(0 \mid 3\tau/2)} = \frac{3i}{2c(q)}. \tag{2.15}$$

To see this, we need the fact that (See [2, Proposition 2.2])

$$c(q) = 3q^{1/3}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3}{(1-q^n)}.$$

Since $\vartheta_1(z \mid \tau) = -iq^{1/4}e^{iz}\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}e^{2iz})(1-q^{2n-2}e^{-2iz})$ and $\vartheta_1'(0 \mid \tau) = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})^3$ (see [5, p. 470])

$$\begin{aligned}\frac{\vartheta_1(\pi\tau \mid 3\tau/2)}{\vartheta_1'(0 \mid 3\tau/2)} &= \frac{i}{2}q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^{3n-1})(1-q^{3n-2})}{(1-q^{3n})^2}\\ &= \frac{i}{2}q^{-1}\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{3n})^3}.\end{aligned}$$

Clearly, this and the above product formula of $c(q)$ yield (2.15).

We now establish (2.14). We note that after substituting (2.13) into (2.14), it reduces to the proof of the identity:

$$-a(q^{2/3})+4 = 3i\frac{\vartheta_1'}{\vartheta_1}\left(\frac{2\pi\tau}{3}\,\middle|\,\tau\right)$$

or

$$-a(q)+4 = 3i\frac{\vartheta_1'}{\vartheta_1}\left(\pi\tau\,\middle|\,\frac{3\tau}{2}\right). \tag{2.16}$$

To prove this, we recall [3] that

$$a(q) = 1 + 6 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^n + q^{2n}} \tag{2.17}$$

And from the identity $\vartheta_4(z + \pi\tau/2 \mid \tau) = iq^{-1/4} e^{-iz} \vartheta_1(z \mid \tau)$, and choosing $z = 0$ in (2.2) we see that

$$\frac{\vartheta_1'}{\vartheta_1}\left(\pi\tau \,\middle|\, \frac{3\tau}{2}\right) = \frac{\vartheta_4'}{\vartheta_4}\left(\frac{\pi\tau}{4} \,\middle|\, \frac{3\tau}{2}\right) - i = -i\left(1 - 2\sum_{n=1}^{\infty} \frac{q^n}{1 + q^n + q^{2n}}\right). \tag{2.18}$$

Clearly, we now see that (2.16) follows from (2.17) and (2.18).

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